# Transforming from AIPS to Haystack Frame.

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## Introduction

The purpose of this memo is to describe the transformation from the AIPS to the Haystack frame. The Haystack delay $\tau$ between two stations labeled $j$, $k$ at some epoch $t$ can be defined in the following way. At some time $t_j$ the signal is incident at station $j$. At another time $t_k$ the signal is received at station $k$. The delay is defined as:

$$\tau_{j,k}(t_j) \equiv t_k - t_j \qquad (1)$$

Note that this definition does not depend at which station the signal is received.

AIPS gives the delay from sets of VLBI stations to the center of the Earth. We are interested in the delay between the stations. Given the delays, and delay rates at some epoch around two sides of a triangle we can compute the delay around the third side. If we also have the delay acceleration we can compute the delay rates.

## Delay Closure

We start by deriving the formula for the delays. Assume we know the delay (and rates) between station 1 and 2, and stations 2 and 3. What is the delay between station 1 and 3? From equation (1) above, it is obvious that this is:

$$\tau_{1,3}(t_1) \equiv t_3 - t_1 = t_3 - t_2 + t_2 - t_1 = \tau_{1,2}(t_1) + \tau_{2,3}(t_2) \qquad (2)$$

Now, by definition (equation 1, above)

$$t_2 = t_1 + \tau_{1,2}(t_1) \qquad (3)$$

Taking a Taylor series expansion of the last term on the Right Hand Side (RHS) of (2) we find:

$$\tau_{2,3}(t_2) = \tau_{2,3}(t_1) + \frac{d\tau_{2,3}(t_1)}{dt_1}\tau_{1,2}(t_1) + \frac{1}{2!}\frac{d^2\tau_{2,3}(t_1)}{dt_1^2}\tau_{1,2}^2(t_1) + ..... \qquad (4)$$

Physically what is happening is that during the transit time the third station moves. Therefore the delay must be compensated for by the amount the station moves.

The last term and the higher order terms on the RHS of (4) can be neglected. One way of seeing this is noting that for two VLBI stations on the earth, the delay will consist of a constant part (due to the N-S part of the baseline) plus a harmonic part (the E-W part of the baseline) with a 24 hour period:

$$\tau(t) = \tau_0 + \tau_H \sin(\tfrac{2\pi t}{86400s} + \phi) \tag{5}$$

with $t$ measured in seconds. The maximum possible delay occurs when the stations are on opposite sides of the earth. In this case:

$$\tau_{Max} = \tfrac{2R_e}{c} = \tfrac{12756km}{c} = 4.25 \times 10^{10} ps \tag{6}$$

From equation (5) it follows that

$$\left| \tfrac{d^m \tau(t)}{dt^m} \tau^m \right| \le \tau_{Max} \times \left(\tfrac{2\pi\tau_{Max}}{86400s}\right)^m < 4.25 \times 10^{10} ps \times (3.1 \times 10^{-6})^m \tag{7}$$

The left hand side of this (LHS) decreases very rapidly with increasing m, as summarized below:

| Degree (m) | Magnitude |
|---|---|
| 1 | $1.3 \times 10^{5}$ ps |
| 2 | $3.9 \times 10^{-1}$ ps |
| 3 | $1.2 \times 10^{-6}$ ps |

Ignoring all but the first order term, we have:

$$\begin{aligned} \tau_{1,3}(t) &= \tau_{1,2}(t) + \tau_{2,3}(t) + \frac{d\tau_{2,3}(t)}{dt} \tau_{1,2}(t) \\ &= \tau_{2,3}(t) + \tau_{1,2}(t) \times \left(1 + \frac{d\tau_{2,3}(t)}{dt}\right) \end{aligned} \tag{8}$$

As a special case of equation (8), suppose that station 3 is colocated with station 1. Then the LHS of (8) vanishes, and we have, after a little algebra,

$$\tau_{1,2}(t) = -\tau_{2,1}(t)/(1 + \tfrac{d\tau_{2,1}(t)}{dt}) . \tag{9}$$

This useful equation relates the delay measured at one end of the baseline to the delay measured at the other end. An alternate form of this is:

$$(1 + \tfrac{d\tau_{2,1}(t)}{dt}) = -\tau_{2,1}(t)/\tau_{1,2}(t) . \tag{10}$$

Exchanging 1 and 2, and multiply both sides, we find:

$$(1 + \tfrac{d\tau_{2,1}(t)}{dt}) = 1/(1 + \tfrac{d\tau_{1,2}(t)}{dt}). \tag{11}$$

This equation relates the delay rate at the two ends of the baseline.

**Conversion of Delays and Rates**

AIPS gives the delays from the stations to a fictitious station at the center of the earth. This delay is time tagged at the time the signal reaches the center of the earth. Let the subscript *e* denote this station. Then the AIPS delays are:

$$\tau_{j,AIPS}(t) = \tau_{e,j}(t) \tag{12}$$

Suppose that we make the substitution $2 \rightarrow e$ in equation (8). Then we have:

$$\tau_{1,3}(t) = \tau_{e,3}(t) + \tau_{1,e}(t) \times \left(1 + \frac{d\tau_{e,3}(t)}{dt}\right) \tag{13}$$

Using equation 9 on the second term we have:

$$\tau_{1,3}(t) = \tau_{e,3}(t) - \tau_{e,1}(t) \times \frac{1 + \frac{d\tau_{e,3}(t)}{dt}}{1 + \frac{d\tau_{e,1}(t)}{dt}} \tag{14}$$

The LHS is what we are after, while the RHS is a function only of the AIPS delays and rates. The present version of CL2HF ignores the term in [], or more precisely, sets it to 1. From equation (7) above, this is equivalent to an error of $3 \times 10^{-6}$ which is not negligible.

The equation for the rates can be found by differentiating equation (14). We find:

$$\begin{aligned} \frac{d\tau_{1,3}(t)}{dt} &= \left(\frac{d\tau_{e,3}(t)}{dt} - \frac{d\tau_{e,1}(t)}{dt}\right) \times \frac{1}{1 + \frac{d\tau_{e,1}(t)}{dt}} \\ &\quad - \tau_{e,1}(t) \frac{\frac{d^2\tau_{e,3}(t)}{dt^2} \times \left(1 + \frac{d\tau_{e,1}(t)}{dt}\right) - \frac{d^2\tau_{e,1}(t)}{dt^2} \times \left(1 + \frac{d\tau_{e,3}(t)}{dt}\right)}{\left(1 + \frac{d\tau_{e,1}(t)}{dt}\right)^2} \end{aligned} \tag{15}$$

The present version of CL2HF sets the second term to 0. Although not as large as the first term, it is still important.

**Complications Due to Different Epochs and AIPS Table structure.**

There are several things that complicate the above discussion:

1. AIPS divides the delay into two parts. A model delay and a solution (residual) delay.
2. The epoch of the model delay can be determined prior to the experiment.
3. The epoch of the residual depends on the midpoint of the good data.
4. Neither epoch is the Haystack epoch.

The model delay is given as a 6th order polynomial. For comparison purposes with the Haystack data it is most convenient to reference everything to the Haystack epoch. Let the model delay be denoted by the subscript *M*, the solution delay by the subscript *S*, and the total delay by the subscript *T*. Then

$$\tau_{j,T}(t_H) = \tau_{j,M}(t_H - t_M) + \tau_{j,S} + \tfrac{d}{dt}\tau_{j,S} \times (t_H - t_S) \tag{16}$$

In principle there are neglected terms due to the (unmeasured) residual acceleration. In practice these terms are small. These terms would arise, for example, if the station position was in error. For example, if the station position is in error by $L$ meters than the neglected term has magnitude:

$$\tfrac{1}{2}\tfrac{L}{c}\left(2\pi\tfrac{t_H - t_S}{86400s}\right)^2 \tag{17}$$

For a 10 meter error and a 1 minute interpolation this is about 0.5 ps.

The total rate at the Haystack epoch can be found by differentiating (16):

$$\tfrac{d}{dt}\tau_{j,T}(t_H) = \tfrac{d}{dt}\tau_{j,M}(t_H - t_M) + \tfrac{d}{dt}\tau_{j,S} \tag{18}$$

Since the model delay is given as a polynomial, it is straightforward to evaluate the first term on the RHS of this.

The total acceleration, which is necessary to compute the Haystack rates, can be found by differentiating (18):

$$\tfrac{d^2}{dt^2}\tau_{j,T}(t_H) = \tfrac{d^2}{dt^2}\tau_{j,M}(t_H - t_M) \tag{19}$$

**Connection with Dave Gordon's Conversion**

Instead of (14) above, Dave Gordon uses the following equation:

$$\tau_{1,3}(t) = \tau_{e,3}(t - \tau_{e,1}) - \tau_{e,1}(t - \tau_{e,1}) \tag{20}$$

Taking the Taylor series expansion of this, we find:

$$\tau_{1,3}(t) = \tau_{e,3}(t) - \tau_{e,1}(t) + \left(\tfrac{d\tau_{e,1}(t)}{dt} - \tfrac{d\tau_{e,3}(t)}{dt}\right)\tau_{e,1}(t) \tag{21}$$

The first term on the RHS is identical to the first term on the RHS of (14). The second term on the RHS of (14) can be expanded to give:

$$\begin{aligned} -\tau_{e,1}(t) \times \frac{1 + \frac{d\tau_{e,3}(t)}{dt}}{1 + \frac{d\tau_{e,1}(t)}{dt}} &\simeq -\tau_{e,1}(t) \times \left(1 + \frac{d\tau_{e,3}(t)}{dt}\right) \times \left(1 - \frac{d\tau_{e,1}(t)}{dt}\right) \\ &\simeq -\tau_{e,1}(t) \times \left(1 + \frac{d\tau_{e,3}(t)}{dt} - \frac{d\tau_{e,1}(t)}{dt}\right) \end{aligned}$$

where the neglected terms are less than 0.1 ps.  This is just the remaining part of (20), thus proving that Dave Gordon’s formulation and that of this memo are equivalent.